Variation Inequalities in Ergodic Theory

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Abstract

Let (X, Σ, m) denote a probability space and τ an ergodic and measure preserving transformation from X to itself. Consider the standard ergodic averages $Af(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k x)$. These averages converge almost everywhere, and thus it is natural to ask how much they oscillate as the move toward the limit. In particular, consider the operator

$$V_\varrho(A_n f)(x) = \sup_{(n_k)\nearrow} \left(\sum_{k=1}^{\infty} \left| A_{n_k} f(x) - A_{n_{k+1}} f(x) \right|^\varrho \right)^{\frac{1}{\varrho}}$$

where the supremum is taken over all increasing sequences (n_k). For each x, this operator measures the degree of oscillation of the sequence of averages $(A_n f(x))$. Whenever this operator is finite we have a.e. convergence. It can be shown that $V_\varrho(Af)$ is finite a.e. when $f \in L^1(X)$ and $\varrho > 2$. More generally, we can consider other (less standard) averages, and the associated variation. In particular, let U denote an $\mathbb{R}^2$ action and consider averages of the form $P_h f(x) = \frac{1}{h}\int_0^h f(U_{t,t^2}x)\,dt$. In this context the variation operator becomes

$$V_\varrho(Pf)(x) = \sup_{(h_k)\nearrow} \left(\sum_{k=1}^{\infty} \left| P_{h_k} f(x) - P_{h_{k+1}} f(x) \right|^\varrho \right)^{\frac{1}{\varrho}},$$

where now the supremum is taken over all increasing sequences (h_k) of real numbers. It can be shown that this operator will be finite a.e. for all $f \in L^p$, $p > 1$, (when $\varrho > 2$) and hence the averages $P_h f(x)$ converge a.e. as $h \to \infty$. The advantage of the variation operator is that we can (by first proving results for a truncated version) obtain an inequality for which the transfer principle can be applied. Thus we can bring the tools of harmonic analysis directly into play.