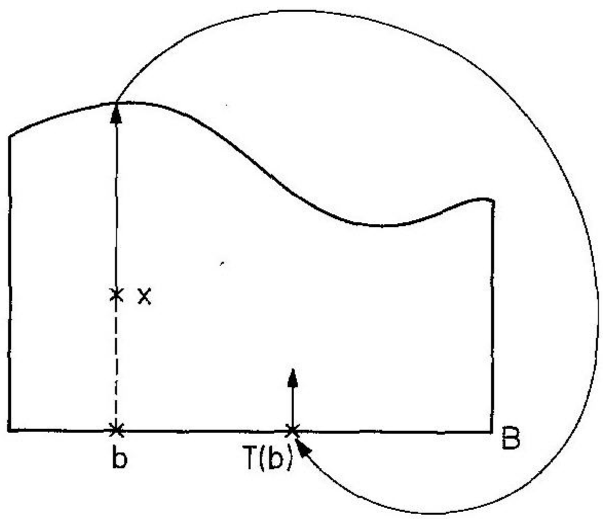
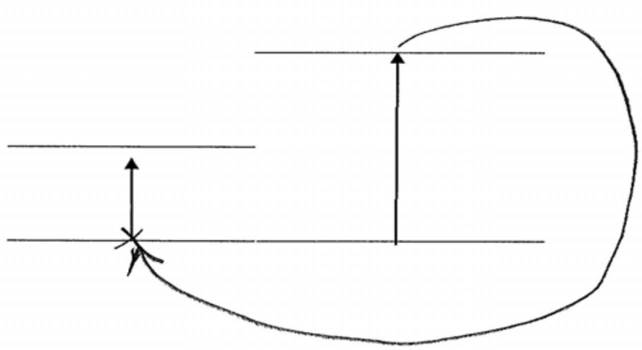
Dan's Thesis

Every flow can be represented as a flow built under a function



Dan



function that takes 2 values

Ergodic Theory Year 1975–1976

Hebrew University, Jerusalem

S. Foguel
H. Furstenberg
S. Goldstein
Y. Katznelson
M. Keane
D. Lind
D. Ornstein
D. Rudolph
G. Schwarz
M. Smorodinsky
J.P. Thouvenot
and
(J. Feldman)

<u>Theorem</u>: If a 2 point extension of a Bernoulli shift is weak mixing, then it is Bernoulli.

A 2 pt extension is:

X is a measure space

T acting on X is Bernoulli

The 2 pt extension acts on the product of X and $\{0, 1\}$

There is a set $E \subset X$

$x, 0 \to Tx, 1$	$x, 1 \to Tx, 0$	x in E
$x, 0 \to Tx, 0$	$x, 1 \to Tx, 1$	x not in E

(i.e., the map is either the identity or the flip on each fiber)

Dan's theorem holds for k point extensions and even compact extensions.

The 2 point extension problem was central.

A major project, based on the then-recent work of Jack Feldman, was to develop a theory parallel to isomorphism theory for equivalence, instead of isomorphism.

2 flows are equivalent if they can be represented as flows built under a function with the same base (cross section) (variable time change)

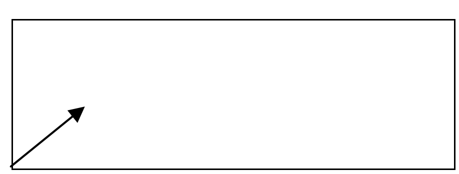
2 transformations are equivalent if they are cross sections of the same flow

(JF): Loosely Bernoulli $\leftrightarrow$ Bernoulli

LB flows, infinite entropy equivalent B flow
infinite entropy

LB flows finite entropy equivalent B flow

LB flows zero entropy equivalent Kroncke flow



Structural Stability (Smale school)

2 flows are "essentially" the same if they are equivalent (and the equivalence doesn't move points very much)

(Poincaré): "reduce" study of flows to study of transformations by taking a cross section

a flow is LB if its cross sections are LB

Another way to reduce flows to transformations is to discretize time

Is a flow LB if its discretizations are LB?

Theorem (Dan): $\Rightarrow$ the answer is yes

Another theorem of Dan's is

if T is LB, then T^2 is LB

Equivalence theory counterexamples

Feldman constructed a non-LB transformation, $\mathcal{J}$

<u>(Dan)</u>: $\mathcal{J}$ and $\mathcal{J}^{-1}$ are not equivalent

<u>(Dan) (based on $\mathcal{J}$)</u>:

∃ uncountably many non-equivalent
transformations of zero, positive,
infinite entropy

Dan's "counterexample machine"
(for isomorphism)

Construct permutations of a finite or countably inifinite set

Theorem (Dan): These lift to mixing transformations

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simplest example:

T_1 and T_2 are not isomorphic, but T_1^2 and T_2^2 are isomorphic.

T_1 comes from the identity on 0,1 and T_2 from the flip

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idea:

construct $\hat{T}$ (acting on X)

take finite or infinite product of X

$X_1 \times X_2 \times X_3 \times \ldots$

$\hat{\hat{T}}$ acts on product :

apply $\hat{T}$ to each factor and then permute

only automorphisms :

apply $\hat{T}^{(i)}$ to each factor, then permute

Examples from the machine

(all T are mixing)

1) T_1 and T_2 not isomorphic
   T_1^n and T_2^n are isomorphic, all $n > 1$

2) T has no roots of any order

3) T has uncountably many 2 pt non-isomorphic factors

4) T has countably many 2 pt factors that are isomorphic, but do not sit in the same way (2 factors of T sit the same way if there is an automorphism of T taking one factor to the other)

Chris Hoffman extended Dan's machine so that the mixing is replaced by K

in particular 1, 2, 3, 4 hold for K

Dan had already proved (1) for K

What kinds of factors can a B shift have?
(as transformations then are all Bernoulli:)

2 factors are the same as factors or sit the same way if there is an automorphism taking one factor to the other

A factor is relatively K if any factor that contains it has greater entropy

A factor is relatively Bernoulli if the whole transformation is the product of the factor and a B shift

Thouvenot initiated the relative study with a relative isomorphism theory for factors that are relatively Bernoulli

Dan is responsible for some of the first relative counter-examples.

He classified the relatively finite (compact) factors of a B shift.

In particular, all factors with 2 pt fibers sit the same way. And only a finite number of ways that a factor with k pt fibers can sit.

In contrast (example 4), he showed that a mixing transformation could have countably many factors with 2 pt fibers that are isomorphic, but do not sit the same way.

Hoffman gave a way of going from Dan's counter-example machine to a relative counter-example machine.

1) F_1 and F_2 do not sit the same way under T but do under T^n $n>1$

2) A relatively K factor that is not invariant under any root of T

3) Uncountably many 2 pt extensions of factors that do not sit the same way as 2 pt extensions (the factor could be relatively K)

Amenable Groups

Dan developed an isomorphism theory for

"actions of a group G, where G can be written as a skew product of Z with some compact metrizable group $\bar{G}$. Thus, G can be written as $\{(n,\bar{g}) | n \in Z, \bar{g} \in \bar{G}\}$ where $(n,\bar{g}) \circ (n',\bar{g}') = (n+n', \phi^{n'}(\bar{g}) \circ \bar{g}')$, ϕ a continuous automorphism of $\bar{G}$. In this case we will write $G = Z \otimes^{\phi} \bar{G}$."

<u>Theorem (Dan)</u>: any 2 actions of $Z \otimes G$, where the Z actions are Bernoulli and have the same entropy are isomorphic.

New phenomena:

The group acting on itself has positive entropy.

Back to classifying the relatively compact (finite) factors of a B shift.

This rests, in part, on Dan's isomorphism theorem for $Z \otimes G$.