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Geometric and Probabilistic Structures in Dynamics

Dedicated to Misha Brin on occasion of his 60th birthday.

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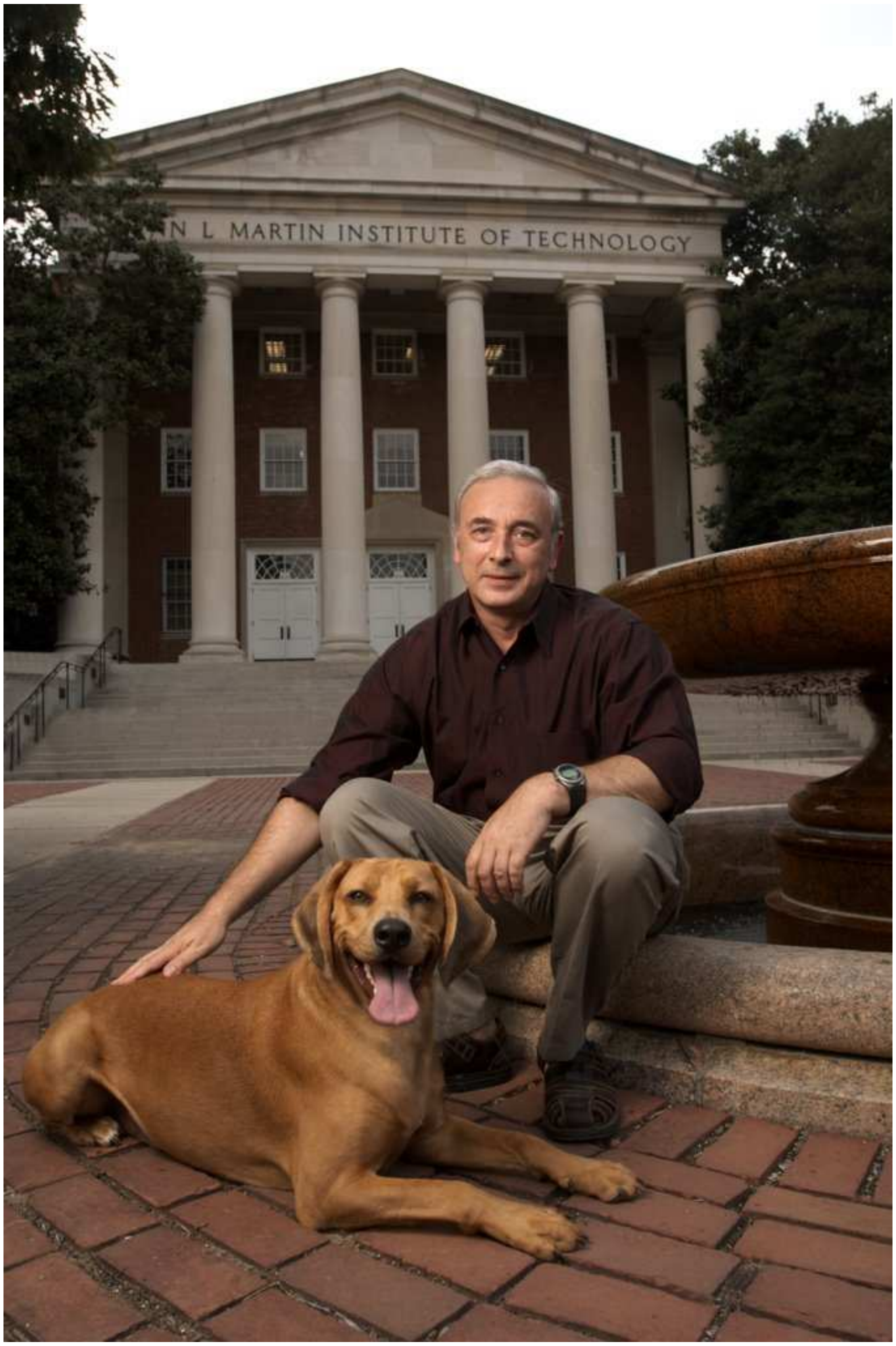
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Preface.

This volume is dedicated to our friend, Misha Brin, on the occasion of his 60th birthday.

Misha's involvement in mathematics began in 1963 when he entered the physical-mathematical school No. 444 in Moscow. The 60s were the golden years of Soviet education in physics and mathematics and many schools specializing in these two disciplines opened in Moscow, Leningrad and some other major cities to attract young boys and girls interested in pursuing career in science. Stimulating creative and independent thinking, these schools gave their students the best possible education in general and in physics and mathematics in particular. For many students these schools were havens sheltering them, to some extent, from the harsh ideology of the ruling Communist Party. In the authoritarian system of the Soviet Union, this could not last too long and indeed the situation gradually change for worse in the end of 60th and beginning of 70 when the government "took care" of many of these schools and institutions of higher education as well (a more detailed account of the events of this time can be found in the article by Katok "Moscow dynamics seminar of the Nineteen Seventies and the early career of Yasha Pesin" that will appear in DCDS, 2008; see also the book "Golden Years of Moscow Mathematics, S. Zdravkovska and P. Duren, editors, History of Mathematics, v. 6, 2007).

Naturally upon graduation students of these special schools would be eager to continue their studies at the best educational institutions. Among them the department of mechanics and mathematics — "Mech-Mat" — of Moscow University was everyone's dream as it was flourishing with many outstanding mathematicians teaching new interesting courses and organizing seminars that attracted many talented young students and essentially became "schools" in certain areas of mathematics. These schools were free to attend but staying there required serious effort and devotion.

Misha entered Mech-Mat in 1965 and two years later joined the seminar ("school") in dynamical systems, which was run by Alexeyev and Sinai. The latter was Misha's undergraduate student advisor.

In 1970 Misha graduated from Moscow university with honors and under normal circumstances would have been admitted into the graduate school — "aspirantura" — but by that time the situation in the department had drastically changed for the worse. Under the new leadership, the Communist Party bureau of the department screened the files of potential candidates and dropped many of them on grounds that had nothing to do with scientific merit. Misha was one of them. Turned down from aspirantura, he ended up working in the State Planning Committee (Gosplan) Economics Institute where his responsibilities had little to do with his education and passion for mathematics. Like many of his schoolmates, who found themselves in a similar situation, he decided, against all the odds, to carry out research in mathematics, but he had to do this after hours. Since he was not associated with any mathematical establishment, routine scientific business like publishing papers, attending conferences, even visiting mathematical libraries at Mech-Mat or the Steklov Mathematical Institute could often be a daunting task. Going abroad to attend an international conference was next to impossible. And defending a PhD, even a very good one, could be an adventure with a sometimes unpredictable outcome.

Fortunately, there were still many seminars run by distinguished mathematicians, which provided the necessary environment for people like Misha to conduct research in mathematics. It was the Anosov-Katok seminar on dynamical systems where Misha was an active member and did most of his work until he left USSR in 1979. While Anosov was Misha's oficial PhD advisor, Katok helped Misha with his research to the extent that Misha considered himself Katok's student. Misha defended his PhD in 1975. Although he was one of the founders of partial hyperbolicity theory and had obtained some great results in the area, he was denied a PhD defense at the most natural place — Moscow University — and had to seek elsewhere for a place that would accept his thesis for defense. With the help of his friends he found one in Kharkov, Ukraine. Misha considered himself lucky, since many of his peers needed many years and much effort to find such a place.

It should be pointed out that for many, who like Misha had to combine research in pure mathematics with their responsibilities in the non-mathematical institutions they worked in, defending their PhD thesis could often mark an end of their activity in mathematics. Not only did working in parallel in two unrelated areas require a lot of time and effort, but enthusiasm and hope, so naturally characteristic of young graduates of prestigious Mech-Mat, were quickly fading with years of hardship and injustice. That is why for some emmigration could provide a way out. However just to declare your intention to emigrate required courage and often was a step into an endeavor with an unpredictable and even dangerous outcome. Misha chose this path and although he was fired from his job, he and his family were fortunate to have to wait only several months for the permission to leave.

After a four month stay at IHES, Misha arrived in the US in October of 1979 and was hired as a visiting professor of mathematics by the University of Maryland, College Park. The next year he became a tenure-track Assistant Professor and in the course of few years he was promoted first to an Associate Professor with tenure and then to Full Professor and he has been working in this department ever since. The move to the West led to a burst of creative activity in Misha's research and several of his crucial contributions to dynamical systems were made during the first five years after his immigration (see below). It also brought to light another of Misha's passions: helping talented mathematicians in their careers. In particular, he helped many Russian mathematicians in their move to US (Ya. P. was one of them) and he gave a lot of assistance to his younger colleagues at Maryland and in the dynamics community (including D.D. and K.B.). Misha was actively involved in the Maryland High School Competitions and in the Westinghouse (now Intel) Science Competition, bringing the spirit of Soviet mathematical Olympiads to US soil.

Misha's devotion to dynamical systems is multifaceted and, besides his research, includes his active role in developing dynamical systems in Maryland (and in particular, co-founding now well recognized Maryland–Penn State semi-annual workshop in dynamical systems and related topics), creating the Brin chair in mathematics and more recently, in establishing the Brin prize in dynamical systems.

Misha's research in mathematics can be characterized by good taste in choosing problems and elegant style. His papers are usually short — this is his style of writing — but many of them provide fundamental contributions in several areas of mathematics and often start new directions of research. Most of his papers combine skillfully the methods of dynamics and geometry.

Partial hyperbolicity is one of Misha's favorite subjects. The modern theory of dynamical systems gives very precise information about several classes of low dimensional systems, but much less is known about the higher dimensional situation. One notable exception is uniformly hyperbolic systems, which are well understood due to the classical work of Smale, Anosov, Sinai, Ruelle, Bowen and others. Partial hyperbolicity theory deals with the systems that uniformly contract and expand some but not necessarily all directions in the tangent space. The idea is to use the methods of uniform hyperbolicity to effectively reduce the dimension of the system. This is currently one of the most effective methods in studying properties of high dimensional systems.

Brin's paper with Pesin [18] lays the foundation for the ergodic theory of partially hyperbolic systems (while the study of geometric properties began earlier in the work of Fenichel and of Hirsch, Pugh and Shub). It establishes the properties of unstable (and stable) foliations, which play a crucial role in studying ergodic properties of partially hyperbolic systems. Among them are absolute continuity and Lipshitz regularity inside center-unstable leaves. The paper also provides the first example of (non-Anosov) stably accessible systems: translations on homogeneous spaces of semisimple Lie groups. The authors prove ergodicity of these systems as well as of the frame flows on manifolds of almost constant negative curvature. (The conditions on curvature were greatly improved in subsequent papers of Brin and his collaborators, see [11, 12].)

The ergodicity results obtained in [18] require strong regularity assumptions on either the stable and unstable or the central distribution. While those assumptions are satisfied in several classical examples, the authors were not able to treat small perturbations of these examples. This has been achieved almost a quarter a century later by Grayson, Pugh and Shub and by Burns and Wilkinson who significantly weakened those assumptions. The current state of the art in this field is that in addition to the crucial accessibility requirement just one technical assumption — pinching of the Mather spectrum — guarantees ergodicity. It is worth mentioning that Misha was a pioneer of the study of systems with pinched spectrum and he obtained several topological results about Anosov diffeomorphisms with pinched spectrum [7, 17].

Building upon the results in [18], Misha investigates compact group extensions of uniformly hyperbolic systems, which are a central example in partial hyperbolicity theory [5, 6]. In [18], he proves that accessibility implies ergodicity and establishes other important statistical properties. He also shows that accessibility is generic within the class of systems he considers. To achieve this he introduces the so-called *quadrilateral argument* that now bears his name. It produces a consecutive increase of the dimension of the accessibility set by taking a homoclinic contour, homotoping it to a point and studying the resulting holonomy. This method remains one of the principle tools for proving genericity of accesibility in partially hyperbolic systems.

In the case the extension is not transitive Misha describes the components of transitivity. Namely, he considers the intersection of the closure of the accessibilty class of an orbit with a fiber and proves that it is an orbit of a group now called the *Brin group*. In [6], he shows that the transitivity class of a point is a leaf of the foliation generated by the stable and unstable foliations and the orbit of the Brin group. Brin groups play a key role in the study of group extensions of hyperbolic systems. For example, Parry and Pollicott extended the results about ergodicity of the frame flows to Gibbs measures. This allows one to prove various counting results for frame flows, which were previously known only in the constant curvature case via Selberg type formulas.

A remarkable application of Misha's work on compact group extensions appears in the proof of existence of a C^∞ Bernoulli diffeomorphism on every compact manifolds, [10]. The study of the interplay between topology and dynamical properties has a long history that goes back to Boltzmann who proposed that in problems of statistical mechanics one can interchange time and space averages. It is ergodic systems for which this is possible. In the 1930s Birkhoff posed the problem whether every manifold carries an ergodic system. This problem was solved by Oxtoby and Ulam, but their construction had two drawbacks. First, their examples were not smooth and, second, they had quite weak statistical properties, for example, zero entropy (even though the authors were not aware of that!). The Bernoulli property, which says that a system is measure theoretically isomorphic to a shift on a space of independent discrete random variables, is the strongest chaotic property the system could possess, and it is only natural to ask if every manifold carries a Bernoulli diffeomorphism. For surfaces an affirmative answer was given by Katok (*Bernoulli diffeomorphism on surfaces*, Ann. Math. v.110 (1979), 529–547). In [10], Brin, Feldman and Katok obtained a complete solution of this problem.

Another paper about skew products is [14], which treats random diffeomorphisms, i.e. skew products over a Bernoulli shift. It turns out that the results about Brin groups stated above can be extended to random diffeomorphisms provided the transition probability of each point has smooth density. These results play a key role in the perturbation theory of Lyapunov exponents for random products. An extension of these results to more general partially hyperbolic systems obtained in works of Avila, Bonatti, Gómez-Mont, Viana and others provides a powerful alternative to the Pugh-Shub approach in proving stable ergodicity.

Recently Misha has returned to the theory of partially hyperbolic systems [8, 9] bringing new insight to some old problems. While the stable and unstable distributions are uniquely integrable, the behaviour of the central, center-stable and center-unstable distributions is much less understood. Previously, unique integrability has been shown locally near classical examples in a work of Hirsch, Pugh and Shub. Misha introduces new methods, which make it possible to establish unique integrability (or non-integrability) for all systems homotopic to a given one. The study of unique integrability of center-stable and center-unstable distributions is one of the ingredients of [9], which shows that there are no partially hyperbolic systems with integrable center on $\mathbb{S}^3$.

Partial hyperbolicity provides one way to investigate exponential instability of trajectories. Entropy theory gives another one. Entropy is one of the most powerful invariants for measure preserving systems. Amazingly, in spite of its measure theoretic nature, it also contains information about geometry of the system. One link between geometry and entropy is given by the local entropy formula [13]. Recall that the Shannon-Breiman-McMillan theorem says that the entropy of an ergodic system can be computed as the exponential decay rate of the dynamical refinements of a generating partition. The paper [13] shows that the same limit can be obtained by taking the decay rate of the measures of dynamically defined balls. The *Brin-Katok local entropy formula* plays an important role in dimensional and multifractal analysis of invariant measures.

Another of Misha's favorite subjects is the geometry of spaces of non-positive curvature. The interplay between the curvature and the global geometry/topology of the underlying space is one of the main subjects of differential geometry. In particular, both manifolds of negative and of zero curvature are well understood. The case of non-positive curvature is much more complicated and much less is known. In [3, 4], Misha with several collaborators made a fundamental contribution to the theory of manifolds of nonpositive curvature. This pioneering work laid foundations of the modern theory of geometric rigidity. One approach to the study of non-positively curved manifolds is to classify the spaces that do not behave like strictly negatively curved manifolds. A key result in this area is the following. Let the rank of a tangent vector be the dimension of the space of parallel Jacobi fields along the geodesic defined by this vector. Define the rank of a manifold as the minimal rank of all unit tangent vectors. Let M be a compact non-positively curved manifold of rank at least two. Then M is flat, or has a cover that is a product of two non-positively curved manifolds, or is locally symmetric. This result was obtained by Ballmann and by Burns and Spatzier using a crucial result established in [3, 4] that a non-positively curved manifold of rank k has $k-1$ independent first integrals.

This higher rank rigidity theorem plays a fundamental role in the study of compact non-positively curved manifolds. Already [3, 4] contain a host of new results about compact non-positively curved manifolds, for example, density of closed geodesics. An interesting current direction of research is to extend this classification theorem to singular spaces. Two such extensions are obtained in [1] (for two-dimesional non-positively curved polyhedra) and in [2] (for three-dimensional Euclidean polyhedra). Another rigidity theorem is obtained in [15] dealing with the Brownian motion on compact non-positively curved surfaces. The authors prove that the harmonic measure has positive Hausdorff dimension and is singular with respect to the Liouville measure except for the constant curvature case. This subject is continued in [16] where harmonic measures on 2-dimensional Euclidean complexes, which are hyperbolic in the sense of Gromov is studied.

It is remarkable that many subjects in which Misha was one of the pioneers, including ergodic theory of partially hyperbolic systems, rigidity of non-positively curved spaces, and the dynamics of random transformations and Markov processes on singular spaces, are currently "hot topics" in mathematical research. The present volume, containing contributions by Misha's friends, presents a sample of diverse results related to Misha's research interests.

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