Multiple ergodic averages and combinatorics

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Classical results in ergodic theory

Throughout, $(X, \mathcal{X}, \mu, T)$ denotes a *system*, meaning a measure preserving probability system with an invertible measure preserving transformation T.

Poincaré's Recurrence Theorem:
Let $(X, \mathcal{X}, \mu, T)$ be a system and let $A \in \mathcal{X}$ be a set of positive measure. Then

$$\mu(A \cap T^n A) > 0$$

for infinitely many $n \in \mathbb{N}$.

A subset E of the integers $\mathbb{Z}$ is *syndetic* if $\mathbb{Z}$ can be covered by finitely many translates of E.

Khintchine's Recurrence Theorem:
Let $(X, \mathcal{X}, \mu, T)$ be a system, let $A \in \mathcal{X}$ be a set of positive measure and let $\epsilon > 0$. Then

$$\{n \in \mathbb{Z} : \mu(A \cap T^n A) > \mu(A)^2 - \epsilon\}$$

is syndetic.

Multiple Ergodic Theorem (Furstenberg): Let $(X, \mathcal{X}, \mu, T)$ be a system, let $A \in \mathcal{X}$ be a set with $\mu(A) > 0$ and let $\ell \geq 2$ an integer. Then

$$\liminf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^n A \cap T^{2n} A \cap \cdots \cap T^{(\ell-1)n} A) > 0 .$$

In particular, there exist infinitely many integers n such that

$$\mu(A \cap T^n A \cap T^{2n} A \cap \cdots \cap T^{(\ell-1)n} A) > 0 .$$

Poincaré's Theorem is $\ell = 2$.

What is the corresponding generalization of Khintchine's Theorem?

Given an integer $\ell > 1$ and $\epsilon > 0$, is the set

$$\Big\{ n \in \mathbb{Z} \colon \mu\Big(A \cap T^n A \cap \cdots \cap T^{(\ell-1)n} A\Big) > \mu(A)^\ell - \epsilon \Big\}$$

syndetic?

Answer depends on the LENGTH ℓ of the arithmetic progression!

Theorem: Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and let $A \in \mathcal{X}$. Then for every $\epsilon > 0$, the subsets

$$\Big\{n \in \mathbb{Z} : \mu(A \cap T^n A \cap T^{2n} A) > \mu(A)^3 - \epsilon\Big\}$$

and

$$\Big\{n \in \mathbb{Z} : \mu(A \cap T^n A \cap T^{2n} A \cap T^{3n} A) > \mu(A)^4 - \epsilon\Big\}$$

of $\mathbb{Z}$ are syndetic.

For arithmetic progressions of length ≥ 5, the analogous result does not hold.

Theorem: There exists an ergodic system $(X, \mathcal{X}, \mu, T)$ and for all integers $\ell \geq 1$ there exists a set $A = A(\ell) \in \mathcal{X}$ with $\mu(A) > 0$ such that

$$\mu(A \cap T^n A \cap T^{2n} A \cap T^{3n} A \cap T^{4n} A) \leq \mu(A)^\ell / 2$$

for every integer $n \neq 0$.

Combinatorial Consequences

Furstenberg used multiple ecurrence to prove Szemerédi's Theorem.

Szemerédi's Theorem (finite version): For every integer $\ell \geq 1$ and every $\delta > 0$, there exists $N(\ell, \delta)$ so that for all $N > N(\ell, \delta)$, every subset A of $\{1, \ldots, N\}$ with at least δN elements contains an arithmetic progression of length ℓ.

For an arithmetic progression $\{a, a+s, a+2s, \ldots, a+(\ell-1)s\}$, s is the *step* of the progression.

Write $\lfloor x \rfloor$ for integer part of x. From Szemerédi's Theorem, can deduce that every subset E of $\{1, \ldots, N\}$ with at least δN elements contains at least $\lfloor c(k, \delta) N^2 \rfloor$ arithmetic progressions of length k, where $c(k, \delta)$ is some constant. Therefore the set E contains at least $\lfloor c(k, \delta) N \rfloor$ progressions of length k with the same step.

Ergodic results $\Rightarrow$ improvement for $\ell = 3, 4$.

Theorem: For all real numbers $\delta > 0$ and $\epsilon > 0$ and every integer $K > 0$, there exists an integer $M(\delta, \epsilon, K) > 0$ such that for all $N > M(\delta, \epsilon, K)$ and every subset $E \subset \{1, \ldots, N\}$ with $|E| \geq \delta N$ there exist:

- a subinterval J of $\{1, \ldots, N\}$ with length K and an integer $s > 0$ such that

$$\Big| E \cap (E - s) \cap (E - 2s) \cap J \Big| \geq (1 - \epsilon)\delta^3 K \ .$$

- a subinterval J' of $\{1, \ldots, N\}$ with length K and an integer $s' > 0$ such that

$$\Big| E \cap (E - s') \cap (E - 2s') \cap (E - 3s') \cap J \Big| \geq (1 - \epsilon)\delta^4 K \ .$$

Similar bound for longer progressions does not hold. Proof does not use ergodic theory.

Nonergodic counterexample

Ergodicity is not needed for Khintchine's Theorem, but is essential for ours.

Theorem: There exists a (nonergodic) system (X, μ, T) and, for every integer $k \geq 1$, there exists a subset A of X of positive measure so that

$$\mu\Big(A \cap T^n A \cap T^{2n} A\Big) \leq \frac{1}{2}\mu(A)^k \ .$$

for every nonzero integer n.

Actually there exists a set A of arbitrarily small positive measure with

$$\mu\Big(A \cap T^n A \cap T^{2n} A\Big) \leq \mu(A)^{-c \log(\mu(A))}$$

for every integer $n \neq 0$ and some positive universal constant c.

Idea of proof: $X = \mathbb{T}\times\mathbb{T}$, with Haar measure $\mu = m \times m$ and transformation T given by $T(x,y) = (x, y+x)$.

Let $E \subset \{0,1,\dots,L-1\}$, not containing any nontrivial arithmetic progression of length 3. Define

$$B = \bigcup_{j\in E}\Big[\frac{j}{2L}, \frac{j}{2L} + \frac{1}{4L}\Big) \ ,$$

which we consider as a subset of the torus and $A = \mathbb{T} \times B$.

For every integer $n \neq 0$ we have $T^n(x,y) = (x, y+nx)$ and

$$\mu\Big(A \cap T^n A \cap T^{2n} A\Big) =$$

$$\iint_{\mathbb{T}\times\mathbb{T}} \mathbf{1}_B(y)\mathbf{1}_B(y+nx)\mathbf{1}_B(y+2nx)\, dm(y)\, dm(x)$$

$$= \iint_{\mathbb{T}\times\mathbb{T}} \mathbf{1}_B(y)\mathbf{1}_B(y+x)\mathbf{1}_B(y+2x)\, dm(y)\, dm(x) \ .$$

Now bound this integral.

We get:

$$\mu\Big(A \cap T^n A \cap T^{2n} A\Big) =$$

$$\iint_{\mathbb{T}\times\mathbb{T}} \mathbf{1}_B(y)\mathbf{1}_B(y+x)\mathbf{1}_B(y+2x)\, dm(x)\, dm(y)$$

$$\leq \frac{m(B)}{4L} \ .$$

Behrend's Theorem: For every integer $L > 0$ there exists a subset E of $\{0, 1, \ldots, L-1\}$ having more than $L \exp(-c\sqrt{\log L})$ elements that does contain any nontrivial arithmetic progression of length 3.

So we can choose E of cardinality on the order of $L \exp(-c\sqrt{\log L})$. By choosing L sufficiently large, computation gives statement.

Counterexamples for longer arithmetic progressions

An *integer polynomial* is a polynomial taking integer values on the integers. When P is a nonconstant integer polynomial of degree ≤ 2, the subset

$$\Big\{P(0), P(1), P(2), P(3), P(4)\Big\}$$

of $\mathbb{Z}$ is called a *quadratic configuration of* 5 *terms*, written QC5 for short.

Any QC5 contains at least 3 distinct elements. An arithmetic progression of length 5 is a QC5, corresponding to a polynomial of degree 1.

Ruzsa's Theorem: For every integer $L > 0$ there exists a subset E of $\{0, 1, \ldots, L-1\}$ having more than $L \exp(-c\sqrt{\log L})$ elements that does not contain any QC5.

Theorem: There exists an ergodic system (X, μ, T) and, for every integer $k \geq 1$, there exists a subset A of X of positive measure such that

$$\mu(A \cap T^n A \cap T^{2n} A \cap T^{3n} A \cap T^{4n} A) \leq \frac{1}{2}\mu(A)^k$$

for every integer $n \neq 0$.

Once again, proof gives estimate $\mu(A)^{-c\log(\mu(A))}$, where c is some positive constant.

Construction: $\mathbb{T}$ is torus with Haar measure m.

Take $X = \mathbb{T} \times \mathbb{T}$ and $\mu = m \times m$. Let $\alpha \in \mathbb{T}$ be an irrational and let T be transformation on X given by

$$T(x, y) = (x + \alpha, y + 2x + \alpha) \ .$$

Counterexample for longer progressions in combinatorics

Theorem: For all integers $k > 0$, there exists $\delta > 0$ such that for infinitely many values of N, there exists a subset A of $\{1, \ldots, N\}$ with $|A| \geq \delta N$ that contains no more than $\frac{1}{2}\delta^k N$ arithmetic progressions of length ≥ 5 with the same step.

Let $E \subset \{0, \ldots, L-1\}$ not containing any QC5 and let $\alpha \notin \mathbb{Q}$. For $N \in \mathbb{N}$, define

$$A_N = \{n < N : n^2\alpha \pmod 1 \in B\}$$

and for $s \in \mathbb{N}$, let

$$\Gamma_{N,s} = \{n : n, n+s, n+2s, n+3s, n+4s \in A_N\} .$$

Show there exists infinitely many $N \in \mathbb{N}$ so that

$$|A_N| \geq \frac{1}{2} L|E|$$

and for every $s \geq 1$,

$$|\Gamma_{N,s}| \leq \frac{3N}{L} .$$

Proof uses Ruzsa's Theorem.

Positive Ergodic Results

Fix an integer $\ell \geq 2$, an ergodic system $(X, \mathcal{X}, \mu, T)$ and a $A \in \mathcal{X}$ of positive measure. Key ingredient is study of the sequence

$$\mu\Big(A \cap T^n A \cap T^{2n} A \cap \cdots \cap T^{(\ell-1)n} A\Big) .$$

More generally, let $f \in L^\infty(\mu)$ be a real valued function, we consider the sequence

$$I_f(\ell, n) := \int f(x) \cdot f(T^n x) \cdot \ldots \cdot f(T^{(\ell-1)n} x) \, d\mu(x) .$$

When $\ell = 2$, the correlation sequence $I_f(2,n)$ is the Fourier transform of some positive measure $\sigma = \sigma_f$ on the torus $\mathbb{T}$:

$$I_f(2,n) = \widehat{\sigma}(n) := \int_{\mathbb{T}} e^{2\pi int}\, d\sigma(t) \ .$$

Decomposing the measure σ into its continuous part σ^c and its discrete part σ^d, can write the sequence $I_f(2,n)$ as the sum of two sequences

$$I_f(2,n) = \widehat{\sigma^c}(n) + \widehat{\sigma^d}(n) \ .$$

The sequence $\{\widehat{\sigma^c}(n)\}$ tends to 0 in density, meaning,

$$\lim_{N\to+\infty} \sup_{M\in\mathbb{Z}} \frac{1}{M} \sum_{n=M}^{M+N-1} |\widehat{\sigma^c(n)}| = 0 \ .$$

Equivalently, for any $\epsilon > 0$, the set $\{n \in \mathbb{Z} : |\widehat{\sigma^c(n)}| > \epsilon\}$ has upper Banach density zero.

The sequence $\{\widehat{\sigma^d}(n)\}$ is *almost periodic*, meaning that there exists a compact abelian group G, a continuous real valued function ϕ on G, and $a \in G$ such that $\widehat{\sigma^d}(n) = \phi(a^n)$ for all n.

A compact abelian group can be approximated by a compact abelian Lie group. So any almost periodic sequence can be uniformly approximated by an almost periodic sequence arising from a compact abelian Lie group.

We find a similar decomposition for the multicorrelation sequences $I_f(\ell, n)$ for $\ell \geq 3$. The notion of an almost periodic sequence is replaced by that of a nilsequence:

Let $k \geq 2$ be an integer and let $X = G/\Lambda$ be a k-step nilmanifold. Let ϕ be a continuous real (or complex) valued function on G and let $a \in G$ and $e \in X$. The sequence $\{\phi(a^n \cdot e)\}$ is called a *basic k-step nilsequence*. A *k-step nilsequence* is a uniform limit of basic k-step nilsequences.

A 1-step nilsequence is the same as an almost periodic sequence.

An inverse limit of compact abelian Lie groups is a compact group. However an inverse limit of k-step nilmanifolds is not, in general, the homogeneous space of some locally compact group.

The general decomposition result is:

Theorem: Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system, $f \in L^\infty(\mu)$ and $\ell \geq 2$ an integer. The sequence $\{I_f(\ell, n)\}$ is the sum of a sequence tending to zero in density and an $(\ell-1)$-step nilsequence.

Let $\{a_n : n \in \mathbb{Z}\}$ be a bounded sequence of real numbers. The *syndetic supremum* of this sequence is

$$\sup\Big\{c \in \mathbb{R} : \{n \in \mathbb{Z} : a_n > c\} \text{ is syndetic }\Big\}\ .$$

Every nilsequence $\{a_n\}$ is uniformly recurrent. In particular, if $S = \sup(a_n)$ and $\epsilon > 0$, then set of integers n such that $a_n \geq S - \epsilon$ is syndetic.

If $\{a_n\}$ and $\{b_n\}$ are two sequences of real numbers so that $a_n - b_n$ tends to 0 in density, then the two sequences have the same seyndetic supremum.

Therefore the syndetic supremums of the sequences

$$\{\mu(A \cap T^n A \cap T^{2n} A)\}$$

and

$$\{\mu(A \cap T^n A \cap T^{2n} A \cap T^{3n} A)\}$$

are equal to the supremum of the associated nilsequences, and we are reduced to showing that they are greater or equal to $\mu(A)^3$ and $\mu(A)^4$, respectively.