# Weak Gibbs measures for potentials of weak bounded variation and subexponential instability

Michiko Yuri
Department of Business Administration, Sapporo University
Nishioka, Toyohira-ku, Sapporo 062, Japan

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## 1 Abstract

In this talk, *weak Gibbs measures* for piecewise $C^0$-invertible Markov systems associated to potentials of *weak bounded variation* (WBV) are discussed.

Thermodynamic formalism was satisfactory established for hyperbolic systems along Bowen's program. More precisely, the existence of (finite) generating Markov partitions and analysis of Ruelle-Perron-Frobenius operators associated to Hölder potentials allow one to show the existence of unique equilibrium states which satisfy the Gibbs property (in the sense of Bowen). Also it is well-known that the Pressure functions are analytic and there is no possibility of *phase transition.* All such results require severely *exponential instability* of the dynamics.

Recently, a version of the Gibbs property (the so-called *weak Gibbs*) was proposed in [5] (c.f.[6-9]), which is more suitable to certain nonhyperbolic systems which show *subexponential instability* in some sense. In fact, the weak Gibbs property can be verified for SBR measures invariant under *intermittent maps* i.e., piecewise $C^1$-smooth (finite or countable) Markov maps $T$ defined on bounded regions $X \subset \mathbf{R^d}$ which admit indifferent periodic points (Manneville-Pomeau maps, Brun's map, Inhomogeneous Diophantine algorithm, a complex continued fraction algorithm etc). Although the potentials $-\log|\det DT|$ satisfy neither summable variation nor the bounded distortion, the WBV property is valid ([7]). The SBR measures are equilibrium states for $-\log|\det DT|$ and both uniqueness of equilibrium states and analyticity of pressure functions may fail ([8,10]). Furthermore, invariant densities of the weak Gibbs measures with respect to the normalized Lebesgue measures are typically unbounded at indifferent periodic points ([7,8]) and such singularities can be connected with subexponential behaviour of the dynamical instability and are closely related to the so-called *Intermittency* which is a common phenomena in transition to turbulence and in other complex systems.

The purpose of this talk is : for piecewise $C^0$-invertible Markov systems (see the definition below) to clarify a class of potentials and a property of periodic points (*indifferent*) which may cause phase transition, subexponential instability and so slow deday of correlations. For this purpose, first we construct conformal measures $\nu$ which can be considered as reference measures describing observable phenomena. Next we show the existence of equilibrium states $\mu$ for potentials of weak bounded variation which is equivalent to the conformal measures $\nu$. Lastly, we establish a version of the local product structure (*weak local product structure*) for ergodic measures which are the invertible extention of the ergodic weak Gibbs measures $\mu$ ([9]).

The WBV property of potentials was first observed in [4] when the weak Bernoulli property was established for SBR measures invariant under higher dimensional maps with indifferent periodic points. The proof was based on Ledrappier's idea in [3] which used the absolutely continuity of some conditional measures of the natural extensions and for this Rohlin's entropy formula played an important role. On the other hand, the weak local product structure for the natural extention of weak Gibbs measures allows one to have a more direct proof of the weak Bernoulli property than the one in [4]. We should remark that under certain condition the weak local product structure for the natural extensions coincides with "*asymptotically almost local product structure* " for hyperbolic measures invariant under $C^{1+\alpha}$ diffeomorphisms introduced by Barreira-Pesin-Schmelling in [1] for solving Eckmann-Ruelle conjecture in dimension theory ([9]).

**Definition** We say that a triple $(T, X, Q = \{X_a\}_{a\in I})$ is a *piecewise $C^0$-invertible system* if $X$ is a compact metric space, $T : X \to X$ is a noninvertible map which is not necessarily continuous, and $Q = \{X_a\}_{a\in I}$ is a countable disjoint partition $Q = \{X_a\}_{a\in I}$ of $X$ such that $\bigcup_{a\in I} intX_a$ is dense in $X$ and satisfy the following properties.

**(01)** For each $a \in I$ with $intX_a \neq \emptyset, T|_{intX_a} : intX_a \to T(intX_a)$ is a homeomorphism and $(T|_{intX_a})^{-1}$ extends to a homeomorphism $\psi_a$ on $cl(T(intX_a))$.

**(02)** $T(\bigcup_{intX_a=\emptyset} X_a) \subset \bigcup_{intX_a=\emptyset} X_a$.

**(03)** $\{X_a\}_{a\in I}$ generates $\mathcal{F}$, the sigma algebra of Borel subsets of $X$.

Let a sequence $(a_1 \dots a_n) \in I^n$ satisfy $int(X_{a_1} \cap T^{-1}X_{a_2} \cap \dots T^{-(n-1)}X_{a_n}) \neq \emptyset$. Then we define $X_{a_1\dots a_n} := X_{a_1} \cap T^{-1}X_{a_2} \cap \dots T^{-(n-1)}X_{a_n}$ which is called a cylinder of rank $n$.

**Definition** We say that $\phi$ is a potential of *weak bounded variation* if there exists a sequence of positive numbers $\{C_n\}_{n\geq 1}$ satisfying $\lim_{n\to\infty}(1/n)$

$\log C_n = 0$ and $\forall n \geq 1, \forall X_{a_1 \ldots a_n} \in \bigvee_{i=0}^{n-1} T^{-i} Q$,

$$\frac{\sup_{x \in X_{a_1 \ldots a_n}} \exp(\sum_{i=0}^{n-1} \phi(T^i x))}{\inf_{x \in X_{a_1 \ldots a_n}} \exp(\sum_{i=0}^{n-1} \phi(T^i x))} \leq C_n.$$

**Definition.** A Borel probability measure $\nu$ is called a *weak Gibbs measure* for $\phi$ with a constant $-P$ if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n)\log K_n = 0$ such that $\nu$-a.e.$x$,

$$K_n^{-1} \leq \frac{\nu(X_{a_1 \ldots a_n}(x))}{\exp(\sum_{i=0}^{n-1} \phi T^i(x) - nP)} \leq K_n,$$

where $X_{a_1 \ldots a_n}(x)$ denotes the cylinder containing $x$.

**Definition** Let $\nu$ be an $\exp[P_{\text{top}}(T,\phi) - \phi]$-conformal measure. We say that a periodic point $x_0$ with period $q$ is an *indifferent periodic point* with respect to $\nu$ if $P_{\text{top}}(T,\phi) = \frac{1}{q}\sum_{i=0}^{q-1} \phi T^i(x_0)$ (equivalently $\frac{d(\nu T^q)|_{X_{a_1 \ldots a_q(x_0)}}}{d\nu|_{X_{a_1 \ldots a_q(x_0)}}}(x_0) = 1$).

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